Slice knots & knot concordance
imsundberg.github.io/concordance

Apr 4, 2023

Logistics: currently being streamed on zoom, + recorded
- no audience shown ← messed this up, online audience was visible
- behind password

Who are we?

Arn	she/her	presumably irrelevant, but feel free to duzen us in German
Isaac	they/them	
Hyemhee (Haeli)	she/her	

Who are you? Core target audience: masters + adv. bachelor + PhD

Also welcome: postdocs, other researchers (but note core audience)

i.e. students: you are the target audience so ask us questions

Where are you? Primarily attending in-person
- lectures streamed/recorded to accommodate illness/travel/...

Small group of non-Bonn students (researchers
- encouraged to participate online, incl. HWs.

What do you know? Point-set topology, definition of sm/top manifold

Basic alg. topology: π_1, covering spaces, homology, cohom. Pe'duality, UCT.

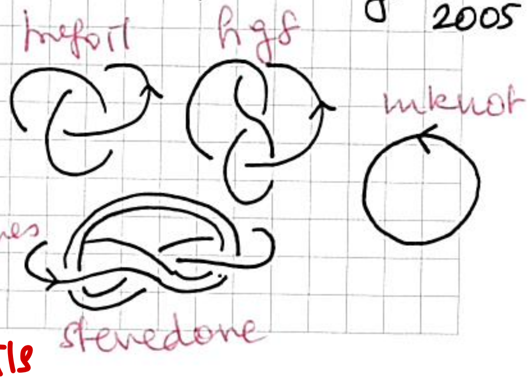
Classical knot theory will be recalled as needed.

What will we do? (See list of topics on course website)

Closest analogue: A survey of classical knot concordance, Livingston 2005

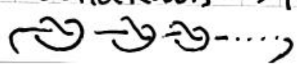
[Today will be necessarily sketchy]

A knot is an embedding $S^1 \hookrightarrow S^3$

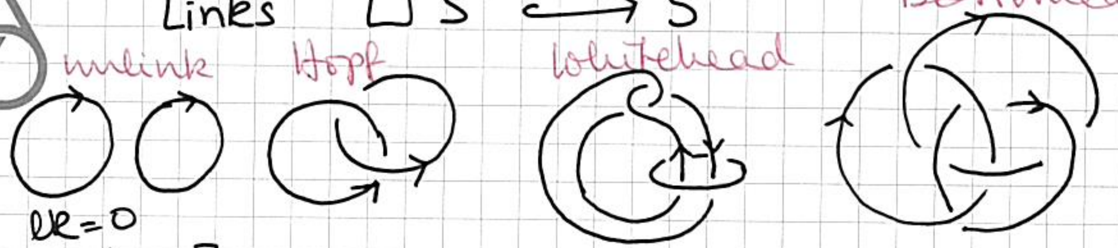


"category" for us $\begin{cases} \text{smooth} \\ \text{piecewise-linear} \\ \text{top. loc. flat} \end{cases}$ can matter, will for us sometimes

wild knots ⟶ top (i.e. homeo onto image)

 ← infinite conn. sum of trefoils

Links $\sqcup S^1 \hookrightarrow S^3$



$lk=0$ [Gauss '1833] $\quad lk=1 \quad lk=0 \quad lk=(0,0,0)$

More generally, $\bigsqcup_i S^{k_i} \hookrightarrow S^n$
ordered, oriented.

Warning: subsets vs maps

Which codimensions?

Codim one: Schoenflies problem [open in DIFF, 4D]

all four in Annals

Zeeman '63 + Stallings '63: Nontrivial PL, TOP knotting only in codim 2 except for smooth

Haefliger '62, Levine '65 } $\exists$ knotted smooth $(4k-1)$-spheres in S^{6k}

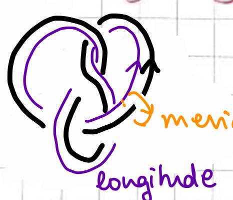
e.g. knotted $S^3 \xhookrightarrow{sm} S^6$

more on high D knot theory, see Ranicki's book

$K: S^k \hookrightarrow S^n$ unknotted if
$$\begin{tikzcd} S^k \arrow[r, hook] \arrow[d, hook, "\partial"'] & S^n \\ D^{k+1} \arrow[ur, "\exists"'] & \end{tikzcd}$$
in the approp. category

There is a lot to say abt classical knot theory, but we will not go into it; rather, we will recall when needed.

One highlight though:



$lk(\text{long}, K)=0$

Dehn surgery on a knot:

Input: $K \subseteq S^3$ knot, $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$

Note: each $\frac{p}{q}$ corr. to a simple closed curve on a torus $S^1 \times S^1$

"p times along meridian + q times along long".

So $\infty = \frac{1}{0} \rightsquigarrow$ meridian

$0 = \frac{0}{1} \rightsquigarrow$ long.

$$S^3_{\frac{p}{q}}(K) := (S^3 \setminus \nu K) \cup S^1 \times D^2$$
(p,q) curve $\longleftrightarrow * \times \partial D^2$

Lickorish-Wallace '1960s

Framed link $\xleftrightarrow[\text{Kirby '70s}]{\text{Dehn surgery}}$ Closed, connected, oriented 3-mfld /homeo

~~Framed link~~ isotopy + blow up/down

So, studying knots/links gives a method to study all 3mfld

Artin '1926: Studied $\Sigma : S^2 \hookrightarrow S^4$

s.t. $K = \Sigma \cap S^3$ nontrivial.

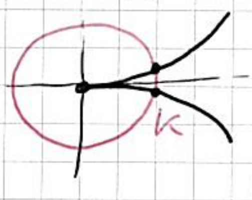
Such a knot is called slice

DIFF, PL, TOP

i.e. $K : S^n \hookrightarrow S^{n+2}$, $\partial \int$ $D^{n+1} \hookrightarrow D^{n+3}$ $\int \partial$

Question: Are all knots slice? w. approp. category

e.g. consider singularities such as



$x^2 + y^3 = 0$ "link of singularity" is the trefoil

[In general, $x^p + y^q = 0$ gives the (p,q) torus knot]

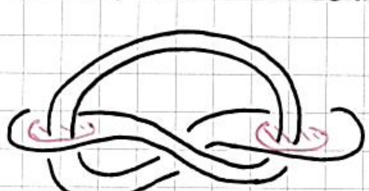
Brauner '1928

Can we replace the singularity by a (sm?) disc?

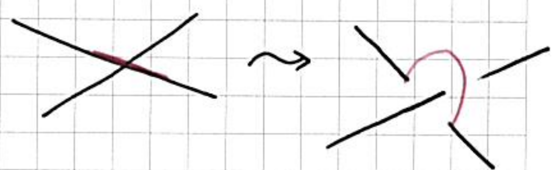
[Kervaire '1965]: All even-dim. knots are slice.

Answer: Murasugi '1960s + Fox-Milnor No. [using signature or Alex poly]

E.g.



analogy:



Warning: Coning

$\text{cone}(K) \subseteq \text{cone}(S^3) = D^4$

is a top. emb. of a disc,

but it is not locally flat

$(\mathbb{R}^n, \mathbb{R}^k) \approx$

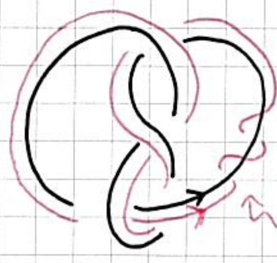
3-mflds $\leftrightsquigarrow$ classical knot theory

similarly 4-mflds $\leftrightsquigarrow$ slice knots.

skipped these both

Knot trace $X_n(K) := B^4 \cup D^2 \times D^2$ ← thickened 2-cell

n-framed neighb. of K $\longrightarrow \partial D^2 \times D^2$



← framing -2

<u>Trace embedding lemma</u>: K $\begin{smallmatrix}\text{sm}\\ \text{TOP}\end{smallmatrix}$ slice $\Leftrightarrow X_0(K) \underset{\text{TOP}}{\overset{\text{sm}}{\hookrightarrow}} S^4$.

Proof (skip?) HW? $(\Longrightarrow)$

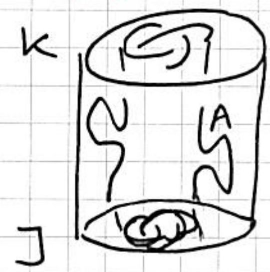
$(\Longleftarrow)$

Palais' 1960

Any two embedding $B^4 \hookrightarrow S^4$ are isotopic.

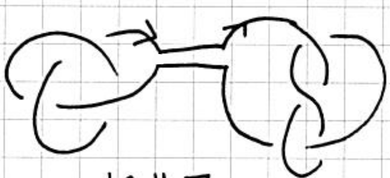
[Key tool in Piccirillo's proof: Conway knot not sm. slice]

Diff. from classical case: Group str.



$K \underset{\text{sm/TOP}}{\simeq} J$, or K & J are $\begin{smallmatrix}\text{sm}\\ \text{TOP}\end{smallmatrix}$ <u>concordant</u>

if they cobound an sm/top annulus in $S^3 \times [0,1]$



$K \# J$

$\left(\text{Knots}/_{\text{sm or top conc}}, \#\right) =: \mathcal{C}^{\text{sm/top}}$

concordance gps.

$[K] = 0 \in \mathcal{C}^{\text{sm/top}}$

$\Longleftrightarrow K$ is $\begin{smallmatrix}\text{sm}\\ \text{top}\end{smallmatrix}$ slice

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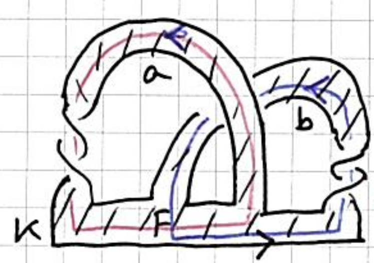
skipped

First obstructions to sliceness



← or

Every K has a <u>Seifert surface</u> $F \subseteq S^3$

Seifert form $H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$

$(a,b) \longmapsto \mathrm{lk}(a, b^+)$

Seifert matrix $\begin{array}{c|cc} & a & b^+ \\ \hline a & -1 & 1 \\ b & 0 & n \end{array} =: V$

$$\Delta_K(t) := \det(V - tV^T)$$

Alexander polynomial

skipped

$$\mathcal{C}^{top} \xrightarrow{\ \phi\ } \mathcal{G} := \left\{ \text{square integral matrices } V^{2g} \text{ s.t. } \det(V - V^T) = \pm 1 \right\} \Big/ \text{"cobordism" of matrices}$$

$$K \longmapsto [V]$$

Seifert matrix.

$$(0 \in \mathcal{G}) := \left[\begin{array}{c|c} 0 & A \\ \hline B & C \end{array}\right]$$

K <u>alg</u> <u>slice</u> if $\phi(K) = 0$

Levine '1969 : $\mathcal{G} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}/2^\infty \oplus \mathbb{Z}/4^\infty$

: $\mathcal{C}_{2n-1} \xrightarrow{\ \phi_n\ } \mathcal{G}_{(-1)^n}$ ⟸ note, only two possibilities, n not that important

odd!

high D concordance purely algebraic

- ϕ_n iso for $n \geq 3$
- ϕ_2 iso onto index 2 subgp
- ϕ_1 is surjective.

Question: Alg slice $\Rightarrow$ slice for 1-knots?

Answer: No [Casson-Gordon '1975]

Definitely no [Cochran-Orr-Teichner '2003] Solvable filtration

$$\mathcal{C} \supseteq \mathcal{F}_0 \supseteq \mathcal{F}_{0.5} \supseteq \mathcal{F}_1 \supseteq \mathcal{F}_{1.5} \supseteq \cdots \supseteq \bigcap_{n \in \frac{1}{2}\mathbb{Z}} \mathcal{F}_n \supseteq \mathcal{T} := \left\{ \begin{array}{l} \text{sm conc. of} \\ \text{TOP slice} \end{array} \right\}$$

$\mathcal{F}_0 = \{\text{alg slice}\}$, $\mathcal{F}_{1.5} \subseteq \{\text{CG invts vanish}\}$

who lives there?

COT, Cochran-Harvey-Leidy '09 : $\mathcal{F}_n / \mathcal{F}_{n.5} \supseteq \mathbb{Z}^\infty \quad \forall n.$

forgot to mention them in lecture, sorry!

using $L^{(2)}$-invts cf Wolfgang Lück

Freedman + Quinn '1980's
Garoufalidis-Teichner '2004 : $\Delta_K(t) = 1 \Rightarrow K$ TOP slice

$$\Downarrow$$

$$K \in \bigcap_{n \in \frac{1}{2}\mathbb{Z}} \mathcal{F}_n$$

+ Donaldson : $\exists$ TOP slice knots that are not sm. slice.

Combinatorial obstr. to sm slicing:
- contact topology (slice-Bennequin inequality)
- Khovanov homology (s-invt)

cf: Lisa's proof.

Indeed $\mathcal{T} \supseteq \mathcal{B}_0 \supseteq \mathcal{B}_1 \supseteq \cdots$

Bipolar filtration [Cochran-Harvey-Horn'13]

$\mathcal{B}_n/\mathcal{B}_{n+1} \supseteq \mathbb{Z}^\infty \ \forall n$ [Cha-Kim'2021]

using $L^{(2)}$-invts + obstr. from Heegaard-Floer homology.

Plan for course: talk abt all of the above

alg conc., CG, COT, Alex poly, Donaldson, ~~exact~~ slice-Bennequin, Khovanov, Heegaard-Floer.

[Wrap up of summary w. some reasons we like slicing knots/links]

1. Given K TOP slice, not sm slice $\rightsquigarrow$ exotic sm. str. on $\mathbb{R}^4$.
   i.e. not diffeo. to std sm str.

   Note: [Moise'77, Stallings'62] $\mathbb{R}^{n\neq 4}$ has a unique sm. str.

skipped

Proof: K TOP slice $\Rightarrow \phi: X_0(K) \xhookrightarrow{TOP} \mathbb{R}^4 \subseteq S^4$ trace embedding

$\mathbb{R}^4 \setminus \phi(\mathring{X}_0(K))$ connected, noncompact 4-mfld

$\Downarrow$ [Quinn'1982]

$\exists$ sm str. extending the str. on $\partial(\underbrace{\phi(X_0(K))}_{\text{inherited sm. str.}})$

Glue together to get $\mathcal{R}$, a smoothing of $\mathbb{R}^4$.

Note $\mathcal{R} \not\cong_{diff} \mathbb{R}^4_{std}$ since then $X_0(K) \hookrightarrow \mathbb{R}^4_{std}$ sm. emb $\Rightarrow$ K sm slice $\Rightarrow\Leftarrow$

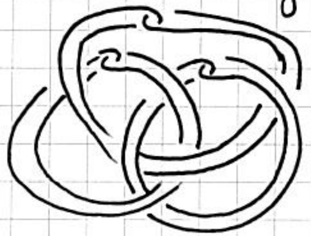
2. " Surgery sequence holds in dimension 4 for all π_1"

$\Updownarrow$

all " good boundary links" are TOP slice with free π_1 of disc comp.

analogue of $\Delta_K(t) = 1$

skipped [e.g.



3. Strategy to disprove sm 4D Poincaré conjecture [Freedman-Gompf-Morrison-Walker '10]

rushed

$K \subseteq S^3$. Find htpy 4-ball W w. $\partial W \cong S^3$

s.t. K sm. slice in W but not in $\mathbb{B}^4$.

Plan: use the s-invt.

Apply to more general 4-mflds? [Manolescu-Piccirillo '21]

How will we do it?

- switch lecturers based on topics.
  next week: Isaac
- problem session, Mondays 10-11, MPIM B27. could be changed as needed
- Discord, Nextcloud, Zoom, pw: sliceknots
- Anonymous feedback form on website
  - all feedback welcome in general.
- Homeworks every week
- Oral exam for Bonn students: definitions, examples, HW problems